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Light-meson orbital excitations in the QCD string approach

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In the framework of the QCD string approach it is shown that the spin-averaged masses $\bar{M}(nL)$ of all low-lying light mesons are well described using the string tension σ as the only parameter. The Regge slope α'_L and the intercept $\alpha_L(0)$ of the Regge L trajectory for $\bar{M}(nL)$ are calculated analytically and turn out to be $\alpha'_L = 0.80 \text{ GeV}^{-2}$ (for $L \leq 4$) and $\alpha_L(0) = -0.34$, in good agreement with the experimental data: $\alpha'_{L \text{ exp}} = 0.81 \pm 0.01 \text{ GeV}^{-2}$, $\alpha_{L \text{ exp}}(0) = -0.30 \pm 0.02$. To obtain this strong agreement with the data the nonperturbative quark self-energy contributions to the meson masses must be taken into account, which appear to be large and negative for small values of L , and are important for a close fit even for larger values of L . From the present analysis of the meson spectra the restriction $\alpha_s \leq 0.40$ on the strong coupling constant is required.

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I. INTRODUCTION

The spectra of hadrons form an extremely important test ground for nonperturbative QCD. The scaling property of QCD tells us that in the end all characteristics of hadrons must depend on a single parameter, say, σ or Λ_{QCD} . Until now, all calculations of hadronic spectra with an accuracy comparable to the uncertainties in the experimental data have relied on models that contain several, in some cases even many, parameters. Here we adopt the formalism that uses the QCD string Hamiltonian which relies on only one parameter, the string tension σ , while, for light mesons, the interactions derived from perturbative QCD, both static and spin dependent, can be considered as a perturbation. The string tension can be extracted from experiment, in particular from the slope of the leading Regge trajectory, and in the present paper we use it to describe the orbital excitations of light mesons.

The QCD string approach developed in recent years [1–3] starts from first principles, i.e., from the Euclidean QCD Lagrangian. In Ref. [2] the relativistic Hamiltonian H_R for the light mesons with spinless quarks was derived under several verifiable assumptions.

First, string excitations (hybrids) are not taken into account, since there is a large gap, $\sim 1 \text{ GeV}$, between a mesonic ground state and its gluonic excitation [4]. Without this approximation one obtains a matrix multichannel Hamiltonian [5]. Therefore the ground states of the mesons with not too large orbital momentum ($L \leq 5$) can be treated in the closed channel approximation.

Second, the relativistic Hamiltonian H_R used here is derived in the quenched approximation, where creation of $q\bar{q}$ pairs (sea quarks) is neglected. The accuracy of this approximation for low-lying states is expected to be approximately 10% [1], while high radial and orbital excitations can be strongly affected by $q\bar{q}$ pair creation [6].

Third, to derive the Hamiltonian H_R only forward-in-time quark trajectories were taken into account while backward

trajectories of a quark (antiquark) were neglected. The accuracy of this approximation was checked by comparing meson spectra of the center-of-mass Hamiltonian H_R and light-cone Hamiltonian H_{LC} , since in the latter case backward-in-time trajectories do not contribute [7]. This comparison has shown that the differences in meson masses for these Hamiltonians are not larger than 10% for all mesons with the exception of the pion, which receives a large contribution from backward-in-time trajectories corresponding to negative energy components, and for them the formalism should be modified. In the Hamiltonian used chiral effects are not taken into account.

Under these assumptions and with the use of the Fock-Feynman-Schwinger (FFS) representation, the Green's function G of a meson with a spinless quark and antiquark can be written in a gauge invariant way as a functional integral with the action A [2]:

$$G = \int_0^\infty ds \int_0^\infty d\bar{s} Dz D\bar{z} \exp(-A),$$

$$A = K + \bar{K} + \sigma S_{\min}, \quad (1.1)$$

where in the action (1.1) the only approximation made is that the vacuum average over the Wilson loop $\langle W(C) \rangle$ is taken in the form of the minimal area law, viz.,

$$\langle W(C) \rangle = \text{const} \times \exp(-\sigma S_{\min}). \quad (1.2)$$

The accuracy of this approximation is determined by the condition

$$R \gg T_g, \quad (1.3)$$

where R is the size (e.g., the rms radius) of the meson and T_g is the gluonic (vacuum) correlation length which determines how the vacuum correlators decrease as a function of the separation r between the quark and the antiquark. The value of T_g was calculated on the lattice [8] and in the quenched case was found equal to $\sim 0.15\text{--}0.20 \text{ fm}$, i.e., it is much

smaller than the rms radii $R(nL)$ for light mesons having $R(nL) \geq 0.8$ fm. Note that the condition (1.3) is also valid for most excited heavy-light mesons and even in heavy quarkonia, where, e.g., $R(1P, c\bar{c}) \sim R(2P, b\bar{b}) \sim 0.65$ fm.

In the general case the approximate area law can be replaced by the exact $q\bar{q}$ interaction which contains a linear confining part for distances satisfying Eq. (1.3) but should be modified at smaller distances, being fully determined by a bilocal field correlator. In the action A (1.1) K and \bar{K} are the quark (antiquark) kinetic energy terms written in the FFS representation:

$$K = \int_0^s \left[m^2 + \frac{1}{4} \dot{z}(\tau)_\alpha^2 \right] d\tau, \\ \bar{K} = \int_0^s \left[m^2 + \frac{1}{4} \dot{\bar{z}}(\tau)_\alpha^2 \right] d\tau, \quad (1.4)$$

where m is the current mass of a quark, τ is the proper time introduced by Schwinger [9], and $z(\tau)$ and $\bar{z}(\tau)$ are the paths of the quark and antiquark.

To define the Hamiltonian one can use the connection between the meson Green's function and the Hamiltonian H :

$$\partial G / \partial T = -H_R G, \quad (1.5)$$

where the Hamiltonian can be determined on any hypersurface, i.e., H can be derived in different frames. Here we shall use the Hamiltonian obtained in the c.m. frame, while in Ref. [7] the Hamiltonian was derived in the light-cone frame.

In order to use the relation (1.5) in Euclidean space-time it is of great importance to go over from the proper time τ to the actual time $t \equiv z_4$ of a quark (antiquark). Doing so the new quantity $\mu(t)$ is introduced:

$$2\mu(t) = \partial t / \partial \tau. \quad (1.6)$$

After performing the canonical quantization the variable μ (being a canonical coordinate) will define the constituent mass of a quark. The last term in the action (1.1) has the form of the Nambu-Goto string:

$$S_{\min} = \int_0^T dt \int_0^1 d\beta \sqrt{\det g}, \quad (1.7)$$

where $g_{ab} = \partial_a w_\alpha \partial_b w^\alpha$ with $a, b = t, \beta$ and $w_\alpha(t, \beta)$ are the coordinates of the string world surface. In Refs. [2] the string was approximated by a straight line connecting the path coordinates $z_\alpha(t)$ and $\bar{z}_\alpha(t)$ and in this case

$$w_\alpha(t, \beta) = z_\alpha(t)\beta + \bar{z}_\alpha(t)(1 - \beta), \quad 0 \leq \beta \leq 1. \quad (1.8)$$

Because of the presence of the square root in S_{\min} in Eq. (1.7) this term cannot be quantized and to get rid of it one can introduce two auxiliary fields $\nu(t, \beta)$ and $\eta(t, \beta)$ in the standard way as is done in string theory [10].

By definition the introduction of the auxiliary fields is accompanied by the additional integration over $D\mu, D\nu, D\eta$ in the functional integral defining the meson Green's function,

$$G = \int D\mu D\nu D\eta DR_\alpha Dr_\alpha \exp(-A), \quad (1.9)$$

where the “center-of-mass” coordinate R_α and the “relative” coordinate r_α are introduced instead of the path coordinates z_α and \bar{z}_α .

As shown in Refs. [2] the integrations over $D\eta$ and DR_α can be performed analytically in the integral (1.9) and after that the Green's function has a simpler form, viz.,

$$G = \int D\mu D\nu Dr \exp(-A_R). \quad (1.10)$$

As the next step instead of performing the integration over $D\mu$ and $D\nu$ in the integral (1.10) one can use an equivalent procedure—to go over to the canonical quantization of the Hamiltonian H_R which corresponds to the action A_R . This Hamiltonian in Minkowski space-time is obtained from A_R in a standard way [2,10] and is taken as a starting point in our analysis [see Eq. (3.1)].

The quantization of H_R has been performed in two cases: the quasiclassical quantization of H_R for large orbital momenta $L, L \gg 1$, in Ref. [11] and for not too large L in Refs. [2]. Here we are interested only in orbital excitations with $L \leq 4$ and in this case the Hamiltonian H_R can be presented as the sum

$$H_R = H_R^{(1)} + \Delta H_{\text{str}}, \quad (1.11)$$

where the term ΔH_{str} is relatively small for $L \leq 4$ and can be considered as a perturbation. Then the problem reduces to the quantization of the simpler Hamiltonian $H_R^{(1)}$ which can be easily done (see Secs. III and IV). At the final stage we obtain the surprising result that after quantization the unperturbed Hamiltonian $\tilde{H}_R^{(1)}$ coincides with the Hamiltonian used in the relativistic potential model (RPM) [12,13]. In this way (for states with $L \leq 4$) the connection between the QCD string Hamiltonian $H_R^{(1)}$ and the RPM is established and we can also calculate the corrections, which are absent in the RPM.

The first correction Δ_{str} , called the string correction, comes from the term ΔH_{str} in Eq. (1.11); it is negative and varying from a value of about -50 MeV for $L=1$ to about -150 MeV for $L=4$.

A second correction to the meson mass, Δ_{SE} , is due to the spin (color magnetic moment) interaction of a quark (antiquark) with the external (vacuum) field when the operator $\sigma_{\alpha\beta} F_{\alpha\beta}$ is inserted in the Wilson loop. The form of this operator is

$$\sigma_{\alpha\beta} = \frac{1}{4} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha). \quad (1.12)$$

TABLE I. The masses $M_0^2(L)$ and $M_0^2(\text{approx})=8L+3\pi\sigma$ for the ground states ($n=0$) with $L \leq 6$ ($\sigma=0.18 \text{ GeV}^2$).

L	0	1	2	3	4	5	6
$M_0^2(nL)$	1.7940	3.2126	4.6431	6.0777	7.5142	8.9518	10.3900
$8L\sigma+3\pi\sigma$	1.696	3.1365	4.5765	6.0165	7.4565	8.8965	10.3365
Difference	5.4%	2.3%	1.4%	1.0%	0.77%	0.62%	0.5%

This nonperturbative self-energy correction was analytically calculated in Ref. [14] with the use of the FFS representation for the quark Green's function. Δ_{SE} is negative and has a rather large magnitude (of the order of -400 MeV to -300 MeV) for all states with $L \leq 4$, slightly decreasing with growing L . Owing to this correction the correct value of the Regge intercept was obtained.

We use here the current quark mass $m=0$ and due to the procedure of canonical quantization the constituent mass of a quark can be defined in a rigorous way as the canonical coordinate μ which is equal to the quark kinetic energy. It is of interest to note that the final expression of the Hamiltonian $\tilde{H}_R^{(1)}$ does not contain the constituent mass at all. However, the constituent mass μ must be defined since it explicitly enters those terms in the Hamiltonian, like the spin-dependent and self-energy terms, which are considered as a perturbation.

Here we consider in detail the spin-averaged meson masses $\bar{M}(nL)$, or the centers of gravity of the nL multiplets (i.e., we neglect the hyperfine and fine-structure splittings) for which the physical picture is simpler and at the same time more universal since the parameters do not depend on spin and isospin.

We concentrate mostly on the orbital excitations with $n=0$ for which experimental data exist for all ground states with $L \leq 5$. Then for the linear confining potential σr all meson masses $\bar{M}(nL)$ can be expressed through a single parameter—the string tension σ . The values of the slope and the intercept of the Regge L trajectory ($L \leq 4$) will be calculated analytically: their numerical values $\alpha'_L = 0.80 \text{ GeV}^{-2}$ ($\sigma=0.18 \text{ GeV}^2$) and $\alpha_L(0) = -0.34$ turn out to be in very good agreement with the experimental numbers. From the Regge slope a restriction on the admissible values of the string tension follows: $\sigma=0.18 \pm 0.005 \text{ GeV}^2$ for the pure linear potential and $\sigma=0.19 \pm 0.01 \text{ GeV}^2$ if the Coulomb interaction is taken into account.

The Coulomb contribution is mainly important for the $1S$ and the $1P$ states having values in the range -200 to -100 MeV , and is considered here in a twofold way: from exact calculations with the linear plus Coulomb potential and also when the Coulomb interaction is considered as a perturbation; both considerations give very close results. For $\sigma=0.19 \text{ GeV}^2$ the QCD coupling is $\alpha_s=0.39$ which is typical for heavy quarkonia, and from our analysis of the meson spectra the following restriction on the strong coupling $\alpha_s \leq 0.42$, is obtained. This number is in good agreement with the two-loop value of the freezing coupling constant obtained in background field theory [15].

II. EXPERIMENTAL DATA

The experimental numbers for the spin-averaged masses $\bar{M}(L)$, or the centers of gravity of the $1L$ multiplet (the ground states with $n=0$), are presented in Table I and need some remarks.

First, all members of the 1^3P_J multiplet are supposed to be known: $a_2(1318)$, $a_1(1235)$, and the $a_0(980)$ too, are considered to form the 1^3P_0 multiplet with $\bar{M}(1P) = 1252 \text{ MeV}$. Similarly, for the $f_J(1P)$ mesons the spin-averaged mass is 1245 MeV [16].

Second, in the case of $L=2$, the fine-structure splittings of the $1D$ -wave mesons are supposed to be suppressed as compared to the P -wave states [17]. As a result, for all members of the $1D_J$ multiplet, e.g., the $\rho_3(1.69)$ and $\pi_2(1.67)$, their masses are very close to each other and one can expect that the true value of $\bar{M}(1D)$ lies between these two values. The same would be valid for the isoscalar mesons, if they were not mixed with other hadronic states, and just this situation is observed in experiment where the masses of the $\omega_3(1.67)$ and the $\omega(1.65)$ have values close to the corresponding isovector mesons [16].

Because of the suppression of the matrix elements (ME) such as $\langle 1/r^3 \rangle$ the spin splittings for the higher orbital excitations such as $1F$, $1G$, etc., should be even smaller than for the $1D$ mesons. Therefore the masses of the $a_4(2.01)$ and the $f_4(2.03)$ are supposed to be close to $\bar{M}(1F)$ and also the masses of the $\rho_5(2.30)$ with $L=4$, and $a_6(2.45)$ and $f_6(2.47)$ with $L=5$, lie close to their centers of gravity. The masses of all orbital excitations ($n=0$) can be nicely described by the Regge L trajectory (see Fig. 1)

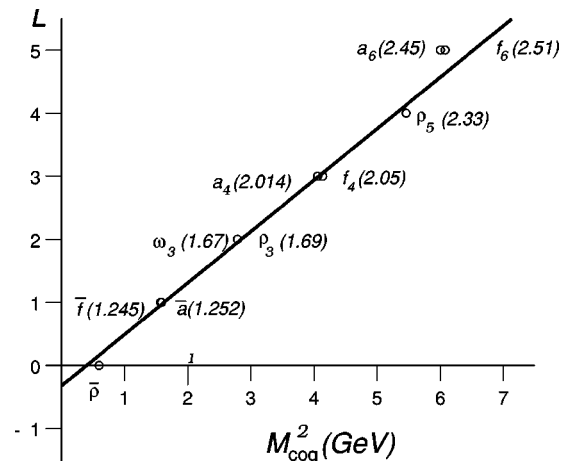


FIG. 1. The Regge L trajectory for the light mesons.

$$\bar{M}^2(L) = (1.23 \pm 0.02)L + 0.37 \pm 0.02 \quad (\text{GeV}^2) \quad (2.1)$$

or

$$L = 0.81\bar{M}^2(L) - 0.30 \quad (2.2)$$

with the following Regge slope and intercept:

$$\alpha'_{L \text{ expt}} = 0.81 \pm 0.01 \quad (\text{GeV}^2)$$

and

$$\alpha_{L \text{ expt}}(0) = -0.30 \pm 0.02 \quad (L \leq 4), \quad (2.3)$$

and both values have a small experimental error.

Note that for the leading ρ trajectory

$$J = \alpha'_J M^2(J) + 0.48 \quad (2.4)$$

the slope $\alpha'_J = 0.88 \text{ GeV}^{-2}$ and the intercept $\alpha_J(0) = 0.48$ are larger since their values depend on the spin contributions. On the contrary, the L trajectory is a universal one and in the approximation of closed channels it is the same for isovector and isosinglet mesons.

In our paper the meson masses, Regge slope, and Regge intercept will be calculated analytically in the framework of the QCD string approach.

III. RELATIVISTIC HAMILTONIAN

We start with the Green's function Eq. (1.10) which is obtained after performing two integrations $D\eta DR$ in the functional integral, so in the Green's function there are still three integrations left: over the two auxiliary fields $\mu(t)$ and $\nu(t)$ and also over the separation r between a quark and an antiquark, $r_\alpha = z_\alpha(t) - \bar{z}_\alpha(t)$. The explicit expression of this action A_R was obtained in Refs. [2] and the dependence of A_R on μ and ν turns out to be rather complicated. Therefore it is more simple and convenient to go over to the equivalent procedure—canonical quantization of the Hamiltonian H_R , which corresponds to the action A_R in Eq. (1.10):

$$\begin{aligned} H_R = & \frac{p_r^2 + m^2}{\mu(\tau)} + \mu(\tau) \\ & + \frac{\vec{L}^2}{r^2} \left[\mu(\tau) + 2 \int_0^1 d\beta \nu(\beta) \left(\beta - \frac{1}{2} \right)^2 \right]^{-1} \\ & + \frac{1}{2} \sigma^2 r^2 \int_0^1 \frac{d\beta}{\nu(\beta)} + \frac{1}{2} \int_0^1 d\beta \nu(\beta). \end{aligned} \quad (3.1)$$

By definition the field operator $\mu(t)$ is

$$\mu(t) = \frac{1}{2} \frac{dt}{d\tau}, \quad (3.2)$$

where t is the actual time.

In Eq. (3.1) m is the current quark mass, which for a light quark (antiquark) will be taken equal to zero; \vec{L} is the orbital

angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, and the operator $p_r^2 = (\vec{p} \cdot \vec{r})^2 / (r^2)$. Canonical quantization of H_R [Eq. (3.1)] has not been done in general, but was performed in two cases: first when the orbital momentum $L \gg 1$ (the quasiclassical quantization in Ref. [11]), and for relatively small $L \leq 4$ which will be considered here. The constant σ determining the nonperturbative potential is the string tension.

It is convenient to rewrite H_R as a sum of two terms:

$$H_R = H_R^{(1)} + \Delta H_{\text{str}}, \quad (3.3)$$

with the “unperturbed” Hamiltonian $H_R^{(1)}$ defined by

$$\begin{aligned} H_R^{(1)} = & \frac{\vec{p}^2 + m^2}{\mu(t)} + \mu(t) + \frac{1}{2} \sigma^2 r^2 \int_0^1 \frac{d\beta}{\nu(\beta)} \\ & + \frac{1}{2} \int_0^1 d\beta \nu(\beta), \end{aligned} \quad (3.4)$$

where in $H_R^{(1)}$ we have included the term $L^2/(\mu r^2)$ and subtracted the same term to give the string correction ΔH_{str} ,

$$\begin{aligned} \Delta H_{\text{str}} = & -\frac{\vec{L}^2}{r^2} \left[\frac{1}{\mu(t)} - \frac{1}{\mu + 2 \int_0^1 d\beta \nu(\beta) \left(\beta - \frac{1}{2} \right)^2} \right] \\ = & -\frac{\vec{L}^2}{\mu r^2} \frac{2 \int_0^1 d\beta \nu(\beta) \left(\beta - \frac{1}{2} \right)^2}{\mu + 2 \int_0^1 d\beta \nu(\beta) \left(\beta - \frac{1}{2} \right)^2}. \end{aligned} \quad (3.5)$$

If L is not large, then the term ΔH_{str} appears to be relatively small and can be considered as a correction to the Hamiltonian $H_R^{(1)}$ [11] but for large L the representation of H_R as the sum Eq. (3.3) is of no use, since in this case both terms are equally important. Note that to get the expression (3.4) one needs the following definition:

$$\vec{p}^2 = p_r^2 + \frac{\vec{L}^2}{r^2}. \quad (3.6)$$

The simplest Hamiltonian H_0 with $L=0$ is a special case of H_R (or $H_R^{(1)}$) with \vec{p}^2 replaced by p_r^2 .

Since ΔH_{str} is considered as a perturbation, the canonical quantization needs to be performed only with the Hamiltonian $H_R^{(1)}$, which can be easily done, since the latter can be presented as the sum of the kinetic energy operator \hat{H}_{kin} which depends on μ and \hat{V}_{pot} which depends on ν :

$$H_R^{(1)} = \hat{H}_{\text{kin}}(\mu) + \hat{V}_{\text{pot}}(\nu). \quad (3.7)$$

The variables μ and ν are the canonical coordinates while the momenta π_μ and π_ν canonically conjugated to μ and ν turn out to be equal to zero, since $H_R^{(1)}$ (as well as H_R) does not depend on the derivatives $\dot{\mu}$ and $\dot{\nu}$. Hence it follows that

$$\dot{\pi}_\mu = \partial H_R^{(1)} / \partial \mu = 0, \quad \dot{\pi}_\nu = \partial H_R^{(1)} / \partial \nu = 0. \quad (3.8)$$

Thus instead of calculating the Green's function as the functional integral Eq. (1.10) and subsequently deriving the Hamiltonian $\tilde{H}_R^{(1)}$, one can obtain this Hamiltonian from $H_R^{(1)}$ [Eq. (3.4)] with the use of the extremum conditions Eq. (3.8). The equivalence of these two procedures was explicitly illustrated in the third paper of Ref. [2] for the kinetic part in Eq. (3.7):

$$\hat{H}_{\text{kin}}(\mu) = \frac{\vec{p}^2 + m^2}{\mu^2} + \mu(t). \quad (3.9)$$

For this Hamiltonian the explicitly calculated Green's function for the free particle is

$$G_0 = \int D\vec{z}(t) D\vec{p} \exp \left[i \int_0^T (\vec{p} \cdot \dot{\vec{z}} - \sqrt{\vec{p}^2 + m^2}) \right] \quad (3.10)$$

which is just the canonical representation of the Green's function with the free Hamiltonian

$$H_0 = \sqrt{\vec{p}^2 + m^2}. \quad (3.11)$$

IV. THE EXTREMAL VALUES OF THE OPERATORS μ AND ν

To understand the physical meaning of the auxiliary fields $\mu(t)$ and $\nu(t)$ let us find their extremal values. First, in the Hamiltonian H_R we determine the variable $\nu(\beta)$ from the extremum conditions (3.8):

$$\frac{\delta H_R^{(1)}}{\delta \nu(\beta)} = 0, \quad \frac{\delta H_R^{(1)}}{\delta \mu(t)} = 0. \quad (4.1)$$

Then one finds that $\nu(\beta)$, which is an operator in general, does not depend on the string parameter β and is equal to

$$\nu_0(\beta) = \sigma r, \quad (4.2)$$

i.e., it is actually the energy density along the string.

With the use of Eq. (4.2) the Hamiltonian $H_R^{(1)}$ reduces to a simpler operator:

$$H_R^{(1)} = \frac{\vec{p}^2 + m^2}{\mu(t)} + \mu(t) + \sigma r, \quad (4.3)$$

where $\mu(t)$ is still an operator in the Hamiltonian formalism. Its extremum can be found from the second extremum condition (4.1):

$$\begin{aligned} \mu(t) &= \sqrt{\vec{p}^2 + m^2}, \quad \text{for } H_R^{(1)}, \\ \mu_0(t) &= \sqrt{p_r^2 + m^2}, \quad \text{for } H_0, \end{aligned} \quad (4.4)$$

i.e., the extremal value of μ is one-half the kinetic energy operator. Note that after canonical quantization the “coordinate” μ is already independent of time. Substituting it into the Hamiltonian $H_R^{(1)}$ one obtains

$$\tilde{H}_R^{(1)} = 2\sqrt{\vec{p}^2 + m^2} + \sigma r, \quad (4.5)$$

giving rise to an eigenvalue equation that is identical to the spinless Salpeter equation (SSE) with a linear potential

$$\tilde{H}_R^{(1)} \psi(nL) = M_0(nL) \psi(nL). \quad (4.6)$$

This equation has been used in the RPM for many years [12,13] and the only difference is that in the Hamiltonian $\tilde{H}_R^{(1)}$ in Eq. (4.6) we must use the current quark mass m since just the current mass enters the meson Green's function ($m=0$ for the light mesons) while in the RPM $m \neq 0$ is usually used, e.g., in Ref. [13] $m = (m_u + m_d)/2 = 220$ MeV.

V. THE CONSTITUENT QUARK MASS

Although the constituent mass μ is not explicitly present in $\tilde{H}_R^{(1)}$, it enters many important physical characteristics like the spin splittings and magnetic moments, and also into the string and self-energy corrections; therefore it must not be left in as an operator. The simplest way to solve this is to define μ_0 as the expectation value of one-half the quark kinetic energy operator Eq. (4.4): i.e.,

$$\mu_0(nL) = \langle \sqrt{\vec{p}^2 + m^2} \rangle_{nL}. \quad (5.1)$$

Note that the eigenvalues $M_0(nL)$ in Eq. (4.6) for the linear potential σr are connected to μ_0 as follows:

$$M_0(nL) = 4\mu_0(nL). \quad (5.2)$$

The values of μ_0 can be expressed through a single parameter—the string tension σ —and the universal numbers $a(nL)$ given by

$$\mu_0(nL) = \sqrt{\sigma} a(nL). \quad (5.3)$$

This relation is a manifestation of the scaling property of the SSE in the case $m=0$.

Another definition of the constituent mass, denoted by $\tilde{\mu}_0$, was used in Refs. [1,3] in the so called “einbein approximation” (EA) where the second extremum condition in Eq. (3.6) is written not for the operator $H_R^{(1)}$ but for the eigenvalues $M_0(nL)$. *A priori* it is not clear whether in both definitions the extremal values $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ coincide or not; therefore let us compare them. In the EA Eq. (4.6) is rewritten as

$$\left[\frac{\vec{p}^2 + m^2}{\tilde{\mu}(t)} + \sigma r \right] \tilde{\psi} = \varepsilon(nL) \tilde{\psi}, \quad (5.4)$$

with

$$\varepsilon(nL) = \left(\frac{\sigma^2}{\tilde{\mu}} \right)^{1/3} A(nL); \quad (5.5)$$

i.e., it reduces to the Airy equation with $\varepsilon(nL) = M_0(nL) - \tilde{\mu}$ and the quantities $A(nL)$ in Eq. (5.5) are the zeros of the Airy function. The constituent mass $\tilde{\mu}_0(nL)$ is now determined by the condition

$$\frac{d\varepsilon(\tilde{\mu})}{d\tilde{\mu}} + 1 = 0 \quad (m=0). \quad (5.6)$$

Then from Eqs. (5.5), (5.6) one obtains that

$$\tilde{\mu}_0(nL) = \sqrt{\sigma} \left(\frac{1}{3} A(nL) \right)^{3/4} = \sqrt{\sigma} \tilde{a}(nL). \quad (5.7)$$

To compare $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ one can use the numbers presented in Appendix A (see Tables VII and VIII) from which the corresponding universal numbers $a(nL)$ and $\tilde{a}(nL)$ can be determined.

The largest difference between $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ was found for S waves and increases with growing radial quantum number n from 5% for the $1S$ state to 7% for the $5S$ state. However, this difference falls with increasing L , being only 1.7% for $L=5$ ($n=0$). So μ_0 and $\tilde{\mu}_0$ are numerically very close. In contrast to the eigenvalues $M_0(nL)$ for the Salpeter and Airy equations a large difference is found between some matrix elements like $\langle 1/r^3 \rangle$ (for any $L \neq 0$) which define the fine-structure splittings. This difference can be as large as 30–50 % in some cases (see Tables VII and VIII below). Moreover, while for the SSE these ME's are growing, they are slightly decreasing for the Airy equation. It is worth noticing that these differences between the ME's would be much larger if a fixed constituent mass, as in potential models, were used.

The reason behind such discrepancies may be connected with the different asymptotic behavior of the wave function (WF). For the SSE (4.5) it falls as $\exp(-\sqrt{\sigma}r)$ [18] while for the Airy equation (5.4) the WF decreases as $\exp(-\sqrt{\mu_0}\sigma r^{3/2})$. Therefore the definition (4.4) of the constituent quark mass as well as the calculations of the ME's with the use of the unperturbed Hamiltonian $\tilde{H}_R^{(1)}$ have to be considered as preferable compared to the EA.

Note a useful relation between the ME's:

$$\langle \sigma r \rangle = 2\mu_0(nL) \quad (5.8)$$

and

$$\langle 1/r \rangle = \sqrt{\sigma} \langle 1/\rho \rangle_{nL}, \quad (5.9)$$

where $\langle 1/\rho \rangle$ is independent of σ but does depend on the quantum numbers.

It is worthwhile to discuss some common features and differences between the QCD string approach used here and the RPM, which was an essential step in our understanding of hadronic spectra.

First of all we have shown that the Hamiltonian used in the RPM is more than a model one, as it can be deduced from the meson Green's function in QCD for not too large angular momenta L , assuming that the string correction coming from the part ΔH_{str} [Eq. (3.5)] is neglected or considered as a perturbation.

However, in the QCD string approach the mass m must be the current quark mass, since just the current mass enters the meson Green's function in the FFS representation. In par-

ticular, for light quarks $m=0$ and for the strange quark $m_s = 140\text{--}160$ MeV are taken [3], while in the RPM $m=(m_s + m_d)/2 = 220$ MeV, $m_s = 465$ MeV are taken in [13].

Since in the QCD string approach the Hamiltonian as well as the string and self-energy corrections are calculated with the use of just the same FFS representation, the whole picture is simplified and the spin-averaged mass can be expressed through the only parameter—the string tension σ .

The constituent mass of a quark does not enter the final form of the Hamiltonian Eq. (4.5) and the notion of the constituent mass appears to be necessary only when one takes into account the spin-dependent interaction and also the string and self-energy corrections to the meson mass. In the FFS representation all these terms are inversely proportional to just the same auxiliary field μ [1] which is strictly determined from the extremum condition due to the procedure of canonical quantization and appears to be the kinetic energy operator. However, to calculate these corrections the variable μ must not be used as an operator and we have defined the constituent mass as the expectation value of the quark kinetic energy. This definition of the constituent mass is in accord with another one—the variational definition of the constituent mass used in Ref. [3].

It is important that the constituent mass in our case depends on the quantum numbers and increases with growing L and n_r .

Finally, instead of the string and self-energy corrections considered in the next sections (they are negative) in the RPM a universal negative constant is introduced.

VI. THE STRING CORRECTION AND THE SLOPE OF THE REGGE TRAJECTORY

It is known that for the Salpeter equation (4.6) (or for the unperturbed Hamiltonian $\tilde{H}_R^{(1)}$) the squared masses $M_0^2(nL)$ can be approximated (with an accuracy of about 1% for $L=0$) by the “string formula” [19],

$$M_0^2(\text{approx}) = 8\sigma L + 4\pi\sigma(n + 3/4). \quad (6.1)$$

The exact values of $M_0^2(nL)$ together with those of $M_0^2(\text{approx})$ ($L \leq 6, n=0$) are given in Table I from which one can see that the differences between them are indeed $\leq 1\%$ for $L \geq 2$.

As is clear from the approximation (6.1), the slope of the Regge trajectory for the SSE is $(8\sigma)^{-1}$, i.e., $\alpha'_L = 0.69 \text{ GeV}^{-2}$ for $\sigma = 0.18 \text{ GeV}^2$, which is 17% smaller than the experimental number Eq. (2.3), $\alpha'_L = 0.81 \pm 0.01 \text{ GeV}^{-2}$. Note that the string corrections which come from the term Eq. (3.5) are also proportional to L and therefore affect the Regge slope. The situation appears to be different in the two domains $L \leq 4$ and $L \geq 5$, respectively, and we consider them separately.

A. Case A: $L \leq 4$

By the definition (3.5) ΔH_{str} gives a negative correction to the eigenvalues $M_0(nL)$; its magnitude turns out to be

TABLE II. The string corrections Δ_{str} in MeV and the mass $M_0(L)$ in GeV, for the ground states ($L \leq 6$).

L	1	2	3	4	5	6
$M_0(L)$	1.7924	2.1549	2.4653	2.7412	2.9920	3.2234
$\Delta_{\text{str}}(L)$	-52.9	-86.9	-113.0	-132.7	-153.7	-170.7
$\Delta_{\text{str}}(\text{asym})^a$	-	-	-	-142.4	-182.9	-219.1

^aFor $L \leq 3$ the asymptotic formula Eq. (6.12) is not applicable.

relatively small, ~ -100 MeV, and therefore this term can be considered as a perturbation [11]:

$$\Delta_{\text{str}}(nL) = \langle \Delta H_{\text{str}} \rangle = - \frac{\sigma L(L+1)}{\mu_0(nL)} \left\langle \frac{1}{r(6\mu_0 + \sigma r)} \right\rangle. \quad (6.2)$$

In Eq. (6.2) we have used that the integral $\int_0^1 d\beta (\beta - 1/2)^2$ is equal to $1/12$ and the operators ν and μ were replaced by their extremal values Eq. (4.2) and Eq. (4.4). The factor in angular brackets can also be approximated (with an accuracy better than 3%) replacing σr by $\langle \sigma r \rangle$. Then the string correction is

$$\Delta_{\text{str}}(nL) = - \frac{\sigma L(L+1) \langle 1/r \rangle}{\mu_0(6\mu_0 + \langle \sigma r \rangle)}. \quad (6.3)$$

Due to the relations (5.2) and (5.8) for the linear potential the correction Δ_{str} becomes

$$\begin{aligned} \Delta_{\text{str}} &= - \frac{\sigma L(L+1) \langle 1/r \rangle}{8\mu_0^2} \\ &= - \frac{2\sigma^{3/2} \langle 1/\rho \rangle L(L+1)}{M_0^2}. \end{aligned} \quad (6.4)$$

Note that in Eq. (6.4) the ME $\langle 1/\rho \rangle \sqrt{L+1}$ is almost constant, varying from 0.787 for $L=1$ to 0.741 for $L=4$ (see Table VII, Appendix A). The values of Δ_{str} (using the ME $\langle 1/r \rangle$ from Tables VIII and IX in Appendix A) are given in Table II.

For comparison in Table II the string corrections valid for large L ($L \geq 5$) [see the asymptotic string correction formula Eq. (6.13)] are also given.

Now one can analytically calculate the Regge slope for the “corrected” mass:

$$M(nL) = M_0(nL) + \Delta_{\text{str}}(nL) \quad (L \leq 4); \quad (6.5)$$

then the squared mass

$$M^2(L) = M_0^2(L) - \frac{4\sigma^{3/2} L(L+1) \langle 1/\rho \rangle}{M_0} + \Delta_{\text{str}}^2. \quad (6.6)$$

If one neglects Δ_{str}^2 in Eq. (6.6), which is small (≤ 0.016 GeV² for $L \leq 4$), and uses the approximation (6.1) for $M_0^2(L)$ then for the orbital excitations with $n=0$ the squared mass Eq. (6.6) becomes

$$\begin{aligned} M^2(L) &= 8\sigma L - \sigma \frac{\sqrt{2} \langle 1/\rho \rangle (L+1)}{\sqrt{L+3\pi/8}} + 3\pi\sigma \\ &= (\alpha'_L)^{-1} L + 3\pi\sigma \end{aligned} \quad (6.7)$$

where the inverse Regge slope in Eq. (6.7) is

$$(\alpha'_L)^{-1} = \left(8 - \frac{\sqrt{2} \langle 1/\rho \rangle (L+1)}{\sqrt{L+3\pi/8}} \right) \sigma = (6.95 \pm 0.02) \sigma. \quad (6.8)$$

The values of $(\alpha'_L)^{-1}$ are essentially constant (see the numbers in Table IX below), varying from the value 6.930σ for $L=1$ to 6.970σ for $L=4$, and we take here $(\alpha'_L)^{-1} = (6.95 \pm 0.02)\sigma$. Then

$$M^2(L) = 6.95\sigma L + 3\pi\sigma \quad (6.9)$$

or

$$L = 0.144 M^2(L) / \sigma - 1.358. \quad (6.10)$$

It gives for $\sigma = 0.18$ GeV² the Regge slope

$$(\alpha'_L)^{-1} = 1.25 \text{ GeV}^2 \quad \text{or} \quad \alpha'_L = 0.80 \text{ GeV}^{-2}, \quad (6.11)$$

in good agreement with the experimental number given in Eq. (2.3) $\alpha'_L(\text{expt}) = 0.81 \pm 0.01 \text{ GeV}^{-2}$. Thus, due to the string corrections we have obtained the correct Regge slope for the spin-averaged masses. However, the intercept in Eq. (6.10) has a very large magnitude and an additional contribution to the meson mass must be taken into account. We discuss this contribution in Sec. VII.

B. Case B: Large L

For large L the extremal value of the operator ν is not equal to σr but turns out to depend on the parameter β as well as on the operator $\mu(r)$. In this case it is a difficult problem to find the exact eigenvalues $M(\text{asym})$ of the Hamiltonian H_R ; therefore in Ref. [11] the eigenvalues of H_R were calculated in the quasiclassical approximation with the following result:

$$M^2(\text{asym}) = 2\pi\sigma \sqrt{L(L+1)} + 3\pi\sigma. \quad (6.12)$$

Here, in the asymptotic mass formula (6.12) the string correction is already taken into account and the constant $3\pi\sigma$ is kept to match the solutions for large L to those for $L \leq 4$. Now, for comparison one can formally define the string correction for large L as the difference between the asymptotic mass Eq. (6.12) and the unperturbed mass $M_0(nL)$ Eq. (5.2):

$$\begin{aligned} \Delta_{\text{str}}(\text{asym}) &= \sqrt{3\pi\sigma + 2\pi\sigma \sqrt{L(L+1)}} - M_0(L) \\ &(L \gg 1, n=0). \end{aligned} \quad (6.13)$$

The asymptotic masses are less than $M_0(L)$ Eq. (4.6) for $L \geq 4$. The magnitude of Δ_{str} is increasing with growing L and for $L = 6$ $\Delta_{\text{str}}(\text{asym})$ is already ≈ 220 MeV.

TABLE III. The squared masses $M^2 = (M_0 + \Delta_{\text{str}})^2$ in GeV^2 for the ground states ($L \leq 6$, $\sigma = 0.18 \text{ GeV}^2$).

L	1	2	3	4	5	6
$(M_0 + \Delta_{\text{str}})^2$	3.026	4.277	5.533	6.794	8.0567	9.319
$M^2(\text{asym})^a$	—	—	—	6.754	7.891	9.026

^aSee the footnote to Table II.

From the numerical values of $M_{\text{str}}(\text{asym})$ (see Table III) one can see that for $L=4$ both string corrections, from the asymptotic formula Eq. (6.13) and from Eq. (6.5), essentially coincide and in what follows the string correction will be taken in the form (6.5) for $L \leq 4$ and from Eq. (6.13) for $L \geq 5$ [when the masses $M(\text{asym})$ are smaller; see Table III].

For $L \gg 1$ the Regge slope in Eq. (6.13) is $(2\pi\sigma)^{-1}$, i.e., for $\sigma = 0.18 \text{ GeV}^2$, $\alpha'_L(L \gg 1) = 0.88 \text{ GeV}^{-2}$ is larger than for $L \leq 4$ and coincides with α'_j for the ρ trajectory. Such a picture is partly seen in experiment, where for $L=5$ the difference $M^2(a_6) - M^2(\rho_5)$ is relatively small and corresponds to the large value $\alpha'_L \approx 1.1 \text{ GeV}^{-2}$. However, this growth of α'_L is likely to be connected with another reason—an effective decrease of the string tension at large distances due to new channels being opened. This effect is considered in our paper [6].

The calculated meson masses (see Table III) still are large compared to experiment and to get agreement between them a negative constant (a fitting parameter) must be added to the squared mass $M^2(nL)$ [13]. Here we shall not introduce a fitting constant, but instead take into account the quark self-energy correction to the meson mass.

VII. THE QUARK SELF-ENERGY CONTRIBUTION AND MESON MASSES

Recently, it was observed that a negative constant must be added to the meson mass, which comes from the nonperturbative quark self-energy contribution created by the color magnetic moment of the quark [14]. This constant is rather large and was calculated with the use of the Fock-Feynman-Schwinger representation of the quark Green's function. The total nonperturbative self-energy contribution, from both the quark and the antiquark, was found to be fully determined by the string tension and by the current mass (flavor) of the quark:

$$\Delta_{\text{SE}}(nL) = -\frac{4\sigma\eta(f)}{\pi\mu_0(nL)}. \quad (7.1)$$

Here $\mu_0(nL)$ is just the constituent mass defined by Eq. (5.1). The constant $\eta(f)$ depends on the flavor: its numerical value for a quark of arbitrary flavor was calculated in Ref. [14]; in particular, for the light mesons we take as in Ref. [14]

$$\eta(n\bar{n}) = 0.90. \quad (7.2)$$

The self-energy terms, as well as the meson masses, are given in Table IV for the ground states ($n=0, L \leq 5$) from

which one can see that $\Delta_{\text{SE}}(L)$ decreases as a function of n and L , being proportional to $\mu_0^{-1}(nL)$. Still, it is rather large (equal to -300 MeV) even for $L=5$.

With the self-energy and the string corrections taken into account the spin-averaged meson mass $\bar{M}(nL)$ is fully determined. The Coulomb correction will be discussed in the next section and calculated in Appendix B.

The meson mass is now given by

$$\bar{M}(nL) = M_0(nL) - \frac{\sigma\langle 1/r \rangle L(L+1)}{\mu_0(6\mu_0 + \langle \sigma r \rangle)} - \frac{4\sigma\eta}{\pi\mu_0} \quad (L \leq 4) \quad (7.3)$$

and for the linear potential can be written as

$$\bar{M}(nL) = M_0(nL) - \frac{2\sigma\langle 1/r \rangle L(L+1)}{M_0^2} - \frac{16\sigma\eta}{\pi M_0}, \quad (7.4)$$

using the relations (5.2) and (5.8). The calculated meson masses ($L \leq 4$) coincide with good accuracy with the experimental values (see Table IV).

For large $L(n=0)$

$$\bar{M} = \sqrt{3\pi\sigma + 2\pi\sigma\sqrt{L(L+1)} + \Delta_{\text{SE}}(L)}. \quad (7.5)$$

VIII. THE INTERCEPT OF THE REGGE TRAJECTORY

From the mass formula (7.4) it follows that the self-energy term enters $\bar{M}(nL)$ in such a way that the negative constant C_0 ,

$$C_0 = -\frac{32\sigma\eta}{\pi}, \quad (8.1)$$

appears in the squared spin-averaged mass $\bar{M}^2(L)$:

$$\bar{M}^2(nL) = (M_0 + \Delta_{\text{str}})^2 - \frac{32\sigma\eta}{\pi} + \left(\frac{16\sigma\eta}{\pi M_0}\right)^2. \quad (8.2)$$

Here the terms Δ_{str}^2 and $\Delta_{\text{str}}\Delta_{\text{SE}}$ will be neglected, because they give small contributions for $L \leq 4$, while the term Δ_{SE}^2 is kept, since it is not small in all states. The constant C_0 is rather large and for $\sigma = 0.18 \text{ GeV}^2$ is equal to -1.65 GeV . Using the expression (6.6) for the mass $(M_0 + \Delta_{\text{str}})^2$ and Eq. (6.1) for M_0^2 , Eq. (8.2) can be presented as

$$\bar{M}^2(L) = (\alpha'_L)^{-1}L + b(L), \quad (8.3)$$

with

$$b(L) = \sigma \left[3\pi - \frac{32\eta}{\pi} + \frac{32\eta^2}{\pi^2(L + 3\pi/8)} \right] \quad (L \neq 0),$$

TABLE IV. The nonperturbative quark self-energy correction $\Delta_{\text{SE}}(L)$ and the meson masses $\bar{M}(L)$ in GeV for the ground states ($\sigma=0.18 \text{ GeV}^2$, $\eta=0.9$).

L	0	1	2	3	4	5
$\Delta_{\text{SE}}(L)$	-0.616	-0.460	-0.383	-0.335	-0.301	-0.294
$\bar{M}(0,L)$	0.723	1.279	1.685	2.017	2.30	2.514 ^a
$\bar{M}_{\text{expt}}(0,L)$	0.612	$1a_J(1.252)$ $1f_J(1.245)$	$\pi_2(1.67)$ $\rho_3(1.69)$ $\omega_3(1.67)$	$a_4(2.014)$ $f_4(2.034)$	ρ_5 2.33 ± 0.04	$a_6(2.45 \pm 0.13)$ $f_6(2.47 \pm 0.050)$

^aThis mass was calculated from the asymptotic formula (6.12).

$$b(L=0) = \sigma \left[3\pi - \frac{32\eta}{\pi} + \frac{256\eta^2}{\pi^2 M_0^2(1S)} \right], \quad (8.4)$$

where for $M_0(L=0)$ it is better to use the exact value, $M_0(1S) = 3.157\sqrt{\sigma}$ and α'_L was already defined by the expression (6.8). From Eq. (8.3) the intercept is

$$\begin{aligned} \alpha_L(0) &= \alpha_L(M^2=0) = -\alpha'_L b(L=0) \\ &= -\frac{b(L=0)}{6.95\sigma}. \end{aligned} \quad (8.5)$$

Note that in $b(L)$ the combination $(3\pi - 32\eta/\pi)\sigma$ is a small number (equal to 0.046 GeV^2 for $\sigma=0.18 \text{ GeV}^2$) and therefore for the intercept the contribution of the self-energy term Δ_{SE}^2 is dominant.

From Eq. (8.4) it is clear that $b(L)$ is sensitive to the value of the flavor factor η , which may introduce an uncertainty on the order of 5%.

With the use of the analytical expression (8.4) and the exact value of $M_0(L=0)$, $\eta(n\bar{n})=0.90$, the quantity $b(L=0)$ is equal to

$$b(L=0) = 2.365\sigma. \quad (8.6)$$

Then the intercept given by Eq. (8.5) takes the value

$$\alpha_L(0) = -(\alpha'_L)b(L=0) = -2.356/6.95 = -0.34. \quad (8.7)$$

This number is in good agreement—larger by only 10%—with the experimental value $\alpha_L(0) = -0.30 \pm 0.02$. It is essential that the intercept does not depend on the string ten-

sion but instead is sensitive to the flavor parameter η . Just for this reason the intercept for the mesons with different flavor depends on the flavor.

So, finally, the Regge L trajectory calculated in the QCD string approach with $\sigma=0.18 \text{ GeV}^2$ is fully determined,

$$L = 0.80\bar{M}^2(L) - 0.34, \quad (8.8)$$

and appears to be very close to Eq. (2.3), obtained from a fit to the experimental spin-averaged meson masses (see Fig. 1). From Eq. (8.8) the averaged mass $\bar{M}(\pi-\rho)$ is found:

$$\bar{M}^2(1S) = 0.425 \text{ GeV}^2 \quad \text{or} \quad \bar{M}(1S) = 0.652 \text{ GeV}, \quad (8.9)$$

which corresponds to a π -meson mass $\bar{M}(\pi) = 301 \text{ MeV}$. This number turns out to be smaller than $M(1S) = 0.723 \text{ GeV}$ calculated directly from Eq. (6.5) and this discrepancy illustrates how sensitive $\bar{M}(1S)$ is to the approximations used.

IX. COULOMB INTERACTION

In the previous sections good agreement of the spin-averaged meson masses (for the ground states with $L \neq 0$) was obtained without taking into account the Coulomb interaction. It is of interest to check whether the Coulomb effects are actually suppressed for $L \geq 0$ states and how large is the Coulomb correction to $\bar{M}(\pi-\rho)$.

To this end we solve the Salpeter equation with the string potential taken as a linear plus Coulomb term, i.e., with the Cornell potential

TABLE V. The spin-averaged masses $\bar{M}_C(L)$ in GeV, theoretical and experimental, for the ground states ($n=0$) ($\sigma=0.19 \text{ GeV}^2$, $\alpha_s=0.39$).

L	0	1	2	3	4	5
$\bar{M}_C(L)$	0.632	1.220	1.650	2.00	2.29	2.51
$\bar{M}_{\text{expt}}(L)$	0.612	$\bar{M}(f_J) = 1.24$ $\bar{M}(a_J) = 1.25$	$\pi_2(1.66)$ $\rho_3(1.69)$ $\omega(1.65)$ $\omega_3(1.67)$	$a_4(2.014)$ $f_4(2.03)$	$\rho_3(2.30)$	$a_6(2.45)$ $f_6(2.47)$

TABLE VI. The exact and perturbative Coulomb corrections $E_C(L)$ (in MeV).

L	0	1	2	3	4	5
$E_C(\text{exact})$	-219	-132	-103	-86.7	-76.2	-68.8
$E_C(\text{pert})$	-194	-126	-99.4	-84.9	-75.1	-68.1
Difference	11.4%	4.5%	3.5%	2.0%	1.4%	1.0%

$$V_C(r) = \sigma r - \frac{4}{3} \frac{\alpha_s}{r}, \quad (9.1)$$

where $\alpha_s = \text{const}$ can be used, since the light mesons have very large sizes, $R \geq 1.0$ fm, and at such distances the strong coupling is saturated and close to the “freezing” value [15]. If for the string tension one takes $\sigma = 0.19$ GeV², then the fitted value of α_s appears to be just the same as for heavy quarkonia [20,21],

$$\alpha_s = 0.39. \quad (9.2)$$

However, the masses of the ground states, including the $1S$ state, can be nicely described with a smaller value for the coupling constant, $0.20 \leq \alpha_s \leq 0.39$, if correspondingly the value of σ is taken from the range $0.18 \text{ GeV}^2 < \sigma \leq 0.19 \text{ GeV}^2$.

The main characteristics of the $q\bar{q}$ system like the eigenvalues $M_C(nL)$ of Eq. (4.6) using the Cornell potential, the constituent masses $\mu_C(nL)$ defined by Eq. (5.1) together with the string and the self-energy corrections are presented in Appendix B in Tables X and XI. Here in Table V we give only the results of our calculations for the spin-averaged masses $\bar{M}_C(nL)$. Note that in the Coulomb case the relation (5.2) is not valid and therefore the meson mass $\bar{M}_C(nL)$ as well as Δ_{str} and Δ_{SE} should be written through the constituent mass (denoted as $\mu_C(nL)$ as in Eq. (7.3) (see Table X where the eigenvalues are given for $\sigma = 0.19 \text{ GeV}^2$, $\eta = 0.90$, and $\alpha_s = 0.39$).

With the use of the string and the self-energy corrections from Table XI the spin-averaged meson masses $\bar{M}_C(L)$, Eq. (7.3), are determined and their values are given in Table V together with the experimental numbers.

TABLE VII. The eigenvalues $M_0(nS)$, constituent masses $\mu_0(nS)$, $\tilde{\mu}_0(nS)$, and matrix elements $\langle 1/r \rangle$ (in GeV) for the Salpeter equation (4.6) and the Airy equation (5.4) with the linear potential σr ($\sigma = 0.18 \text{ GeV}^2$, $L = 0$).

n	0	1	2	3	4
$M_0(nS)$	1.3394	1.9980	2.4985	2.9151	3.2797
$\mu_0(nS)$	0.3348	0.4995	0.6246	0.7289	0.8199
$\tilde{\mu}_0(nS)$	0.3519	0.5351	0.6703	0.7826	0.8807(3)
$\langle 1/r \rangle$ (SSE)	0.3638	0.3299	0.2959	0.2734	0.2559(5)
$\langle 1/r \rangle^a$ (EA)	0.3328	0.2669	0.2334	0.2118	0.1996(5)

^aThese MEs are calculated from Eq. (18) with $\tilde{\mu}_0(nS)$ defined by Eq. (20).

TABLE VIII. The matrix elements $\langle 1/r^3 \rangle$ (GeV³), mass eigenvalues $M_0(nP)$ (GeV), and constituent masses $\mu_0(nP)$ and $\tilde{\mu}_0(nP)$ (GeV) for the P -wave states ($\sigma = 0.18 \text{ GeV}^2$).

n	0	1	2	3	4
$M_0(nP)$	1.7924	2.3153	2.7505	3.1291	3.4682
$\mu_0(nP)$	0.4481	0.5788	0.6876	0.7823	0.8671
$\tilde{\mu}_0(nP)$	0.4620	0.6115	0.7320	0.8335	0.9278
$\langle 1/r^3 \rangle$ (SSE)	0.0264	0.0422	0.0539	0.0635	0.0718
$\langle 1/r^3 \rangle^a$ (EA) = $\tilde{\mu}\sigma/4$	0.0208	0.0275	0.0329	0.0376	0.0417

^aSee the footnote to Table VII.

If now one compares the meson masses $\bar{M}_C(L)$ with those for the linear potential from Table III, one can see that in the Coulomb case for the $1S$ and $1P$ states a better agreement with the experimental numbers is obtained; however, in the Coulomb case the string tension appears to be larger, $\sigma = 0.19 \text{ GeV}^2$. The calculated mass $\bar{M}_C(1S) = 0.632 \text{ GeV}$ is very close to the value Eq. (8.9) from the Regge trajectory Eq. (8.7).

Now the Coulomb correction can be formally defined as the difference between the exact eigenvalues $\bar{M}_C(L)$ and $\bar{M}(L)$:

$$E_C(\text{exact}) = \bar{M}_C(nL) - \bar{M}(nL), \quad (9.3)$$

and compared with the Coulomb corrections $E_C(\text{pert})$:

$$E_C(\text{pert}) = -\frac{4}{3} \alpha_s \langle 1/r \rangle, \quad (9.4)$$

obtained when the Coulomb interaction is considered as a perturbation (see Table VI). In Eq. (9.4) the ME $\langle 1/r \rangle$ is to be taken for the linear potential with the same σ as in the Cornell potential.

The numbers in Table VI demonstrate that the exact and perturbative corrections coincide with an accuracy better than 5% for all states with $L > 0$ (for the $1S$ state the difference is 11%) and therefore these corrections can be calculated as a perturbation.

For the nL states one should also take into account the difference between the exact constituent mass $\mu_C(L)$ and $\mu_0(L)$ for the linear potential; they are related as follows:

$$\mu_C(1L) \approx \mu_0(1L) + |E_C|/3 \quad (n=0),$$

TABLE IX. The constituent masses μ_0 (GeV) and the matrix elements $\langle 1/r^k \rangle$ (GeV^k), ($k = 1, 3$) of the SSE for the ground states ($n=0$) ($\sigma = 0.18 \text{ GeV}^2$, $L \leq 6$).

L	1	2	3	4	5	6
$\mu_0(0L)$	0.4481	0.5387	0.6163	0.6853	0.7480	0.8058
$\langle 1/r \rangle$	0.2362	0.1867	0.1589	0.1406	0.1274	0.1173
$\langle 1/\rho \rangle \sqrt{L+1}$	0.787	0.762	0.742	0.741	0.736	0.732
$\langle 1/r^3 \rangle$	0.0264	0.0098	0.0054	0.0035	0.0026	0.0019

TABLE X. The eigenvalues $M_C(L)$, the constituent masses $\mu_C(L)$, and the ME $\langle 1/r \rangle$ (in GeV) for the SSE with the Cornell potential for the ground states ($n=0$) ($\sigma=0.19$ GeV², $\alpha_s=0.39$).

L	0	1	2	3	4	5
$M_C(L)$	1.157	1.710	2.111	2.446	2.740	3.005
$\mu_C(L)$	0.415	0.496	0.580	0.656	0.710	0.745
$\langle 1/r \rangle_L$	0.484	0.266	0.202	0.170	0.145	0.130

$$\mu_C(nL) \approx \mu_0(nL) + |E_C|/4 \quad (n \neq 0). \quad (9.5)$$

This correction to the constituent mass is mostly important for the $1S$ state. For larger n the difference between μ_C and μ_0 can be neglected. As seen from Table VI, due to the Coulomb interaction all masses are shifted down by an amount in the range of 70 to 200 MeV and therefore a larger value of σ is needed, $\sigma=0.19$ GeV² for $\alpha_s=0.39$, than for the linear potential.

However, one cannot take an arbitrary or too large value for σ , otherwise the Regge slope α'_L would be small and in contradiction with the experimental value. Therefore, in the Coulomb case only values $\sigma=0.19 \pm 0.10$ GeV² are allowed. Then to obtain agreement with experiment using $\sigma \leq 0.20$ GeV² a restriction on the value of the strong coupling constant is found:

$$\alpha_s \leq 0.40 \quad (\sigma \leq 0.20 \text{ GeV}^2); \quad (9.6)$$

otherwise correct numbers for the Regge slope and the intercept cannot be obtained simultaneously.

This upper limit (9.6) for α_s appears to be in accord with the freezing value of the two-loop $\alpha_B(q^2=0)=0.45$ (with the QCD constant $\Lambda^{(3)}=330$ MeV, $N_f=3$) obtained in background field theory [15].

X. CONCLUSIONS

In the framework of the QCD string approach the spin-averaged meson masses with $L \leq 5$ ($n=0$) have been calculated and expressed through a single parameter—the string tension σ and a set of universal numbers. In this approach the kinetic energy is of the same type as in the spinless Salpeter equation. The constituent mass and the nonperturbative quark self-energy are calculable and also depend on the string tension only.

This is the first time accurate predictions for the meson masses have been obtained by relying on one parameter only that is directly connected to the confinement mechanism in QCD.

The analytical expressions for the slope and the intercept of the Regge L trajectory (when the spin splittings are not taken into account) have been deduced, giving rise to a value $\alpha'_L = (6.95\sigma)^{-1} = 0.80$ GeV⁻² ($\sigma=0.18$ GeV²) which coincides with the experimental number. This L trajectory can be considered as a universal one since in the approximation

TABLE XI. The string and self-energy corrections (in GeV) for the SSE with the Cornell potential ($\sigma=0.19$ GeV², $\alpha_s=0.39$).

L	0	1	2	3	4	5
Δ_{str}	0	-0.051	-0.086	-0.112	-0.075	-0.068
Δ_{SE}	-0.525	-0.439	-0.375	-0.332	-0.307	-0.282

of closed channels it does not depend on spin and isospin.

It is shown that the Regge intercept does not depend on σ and $\alpha(M=0)(\text{theory}) = -0.34$ turned out to be only 10% larger than $\alpha(M=0)(\text{expt}) = -0.30 \pm 0.02$. From this intercept $\bar{M}(1S) = 652$ MeV corresponds to a π -meson mass equal to 300 MeV (chiral effects have been neglected here).

For all orbital excitations with $L \neq 0$ the calculated masses are in a good agreement with existing experimental data.

In order to obtain this good agreement with the data we find it necessary to impose a restriction on the value of α_s that is in accord with the freezing picture.

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APPENDIX A: DETAILED SPECTRA

The eigenvalues and the wave functions of the SSE were calculated with the help of the code used before [21,22]. The eigenvalues and relevant matrix elements are given in Tables VII and VIII for the linear potential and in Tables X and XI in Appendix B for the Cornell potential.

From Table VIII one can see that the difference between the ME $\langle 1/r^3 \rangle$ for the SSE and the Airy equations for the P -wave states turn out to be large, reaching 40% for $n \geq 2$. As briefly discussed in Sec. III the reason behind these differences lies in the different asymptotic behaviors of the eigenfunctions of these two equations. In Table IX we also give the constituent masses Eq. (4.6) and the ME $\langle 1/r \rangle$ and $\langle 1/r^3 \rangle$ for the ground states ($n=0$) with $L \leq 6$.

The calculated ME $\langle 1/r \rangle$ is used to obtain the string and Coulomb corrections, while the ME $\langle 1/r^3 \rangle$ can be used to calculate the hyperfine and fine-structure splittings for the mesons with $L \neq 0$.

From Table X one can see that in the Coulomb case the constituent masses $\mu_C(1S)$ and $\mu_C(1P)$ are larger by 29% and 10%, respectively, than for the linear potential (see Table VII) and therefore Δ_{SE} is smaller for them.

APPENDIX B: RESULTS FOR THE CORNELL POTENTIAL

In this appendix we present in Tables X and XI some auxiliary values for the Cornell potential.

- [1] Yu. A. Simonov, *Yad. Fiz.* **60**, 2252 (1997) [*Phys. At. Nucl.* **60**, 2069 (1997)]; in *QCD: Perturbative or Nonperturbative*, Proceedings of the XVII International School of Physics, Lisbon, 1999 (World Scientific, Singapore, 2000), p. 60, hep-ph/9911237.
- [2] A. Yu. Dubin, A. B. Kaidalov, and Yu. A. Simonov, *Phys. Lett. B* **323**, 41 (1994); **343**, 310 (1995); *Yad. Fiz.* **56**, 213 (1993) [*Phys. At. Nucl.* **56**, 1745 (1993)].
- [3] Yu. S. Kalashnikova, A. V. Nefediev, and Yu. A. Simonov, *Phys. Rev. D* **64**, 014037 (2001).
- [4] Yu. S. Kalashnikova and D. S. Kuzmenko, *Phys. At. Nucl.* **64**, 1716 (2001), and references therein.
- [5] Yu. A. Simonov, *Yad. Fiz.* **64**, 1959 (2001) [*Phys. At. Nucl.* **64**, 1876 (2001)].
- [6] A. M. Badalian, B. L. G. Bakker, and Yu. A. Simonov, this issue, *Phys. Rev. D* **66**, 034026 (2002).
- [7] V. L. Morgunov, V. L. Shevchenko, and Yu. A. Simonov, *Phys. Lett. B* **416**, 433 (1998).
- [8] A. DiGiacomo and H. Panagopoulos, *Phys. Lett. B* **285**, 133 (1992); A. DiGiacomo, E. Meggiolaro, and H. Panagopoulos, *Nucl. Phys. B* **483**, 371 (1997); G. S. Bali, N. Brambilla, and A. Vairo, *Phys. Lett. B* **421**, 265 (1998).
- [9] J. Schwinger, *Phys. Rev.* **82**, 664 (1951); Yu. A. Simonov, *Nucl. Phys. B* **307**, 512 (1988).
- [10] E. S. Fradkin and D. M. Gitman, *Phys. Rev. D* **44**, 3230 (1991).
- [11] V. L. Morgunov, A. V. Nefediev, and Yu. A. Simonov, *Phys. Lett. B* **459**, 653 (1999).
- [12] D. P. Stanley and D. Robson, *Phys. Rev. D* **21**, 3180 (1980); *Phys. Rev. Lett.* **45**, 235 (1980); J. L. Basdevant and S. Boukraa, *Z. Phys. C* **28**, 413 (1983).
- [13] S. Godfrey and N. Isgur, *Phys. Rev. D* **32**, 189 (1985).
- [14] Yu. A. Simonov, *Phys. Lett. B* **515**, 137 (2001).
- [15] Yu. A. Simonov, *JETP Lett.* **57**, 525 (1993); A. M. Badalian and D. S. Kuzmenko, *Phys. Rev. D* **65**, 016004 (2002).
- [16] Particle Data Group, D. Groom *et al.*, *Eur. Phys. J. C* **15**, 1 (2000).
- [17] A. M. Badalian, B. L. G. Bakker, and V. L. Morgunov, *Yad. Fiz.* **63**, 1722 (2000) [*Phys. At. Nucl.* **63**, 1635 (2000)].
- [18] D. Duncan, E. Eichten, and H. Thacker, *Phys. Lett. B* **303**, 109 (1993).
- [19] T. J. Allen, G. Goebel, M. G. Olsson, and S. Veseli, *Phys. Rev. D* **64**, 094011 (2001).
- [20] E. Eichten *et al.*, *Phys. Rev. D* **21**, 203 (1980).
- [21] A. M. Badalian and B. L. G. Bakker, *Phys. Rev. D* **62**, 094031 (2000).
- [22] S. Jacobs, M. G. Olsson, and C. Suchyta, *Phys. Rev. D* **33**, 3338 (1986).